C24 Dynamical Systems

Lecture 2: Equilibria and Stability

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Lecture 2: Equilibria and Stability

- Equilibrium definitions
- Stability definitions for equilibrium points
- Phase space of linear systems
- Linearization and the stability of equilibria of nonlinear systems

Equilibria of continuous time systems

An **equilibrium** is a point in state space where $\dot{\mathbf{x}} = 0$:

$$\mathbf{x}^* \text{ is an equilibrium of } \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \text{ if and only if } \mathbf{f}(\mathbf{x}^*) = 0$$

- If $\mathbf{x}(0) = \mathbf{x}^*$, then $\mathbf{x}(t) = \mathbf{x}^*$ for all t
  $\mathbf{x}^*$ is sometimes called a **fixed point**
- For a **linear** autonomous system with non-zero eigenvalues, there is only one solution to $A\mathbf{x}^* = 0$, namely $\mathbf{x}^* = 0$
- In general there may be many points $\mathbf{x}^*$ satisfying $\mathbf{f}(\mathbf{x}^*) = 0$ therefore a **nonlinear** system can have many equilibria

Equilibria of discrete time maps

Discrete time systems also have equilibrium points

$\mathbf{x}^*$ is an equilibrium of $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$ if and only if $\mathbf{f}(\mathbf{x}^*) = \mathbf{x}^*$

- The equilibria of a discrete time system are the **fixed points** of $\mathbf{f}(\mathbf{x})$, so

$$\mathbf{x}_0 = \mathbf{x}^* \quad \Longrightarrow \quad \mathbf{x}_k = \mathbf{x}^* \text{ for all } k$$

- For differential equations, there is a flow of solutions through phase space but the state of a discrete time system 'jumps' between points space, making their trajectories harder to visualise

Flows and equilibria

- We can think of the solution trajectories of a nonlinear ODE as a flow in n-dimensional state space:
  system trajectories are analogous to streamlines in a 3-dimensional fluid

- In this analogy the vector-valued function $\mathbf{f}$ in

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$$

  is a vector field defining the **flow velocity**

- Flows can end or begin at equilibria, or circulate around them

- The stability of the flow near an equilibrium is an important characteristic, which we will focus on today

Stability of flow equilibria

- **Definition**: An equilibrium point $\mathbf{x}^*$ is said to be **stable** if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that all solutions $\mathbf{x}(t)$ satisfy

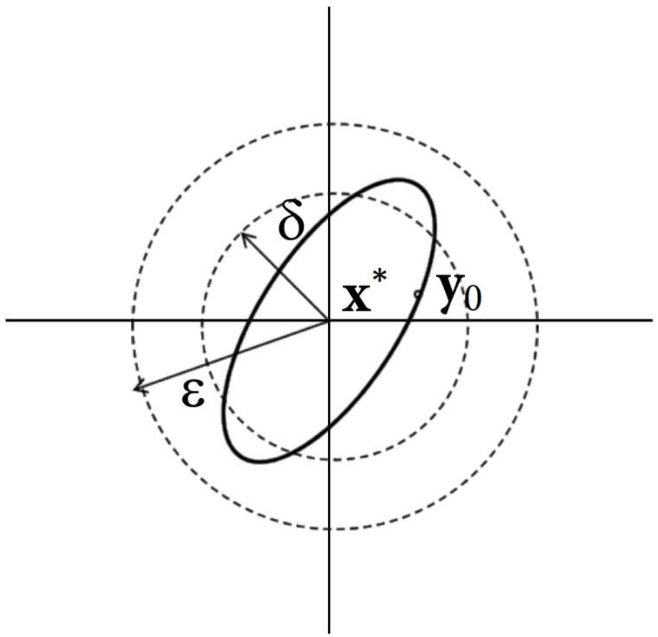
$$\|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon \text{ for all } t \geq 0 \text{ whenever } \|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$$

- Otherwise the equilibrium point is said to be **unstable** (i.e. if, for some $\varepsilon > 0$, no $\delta > 0$ exists satisfying this condition)

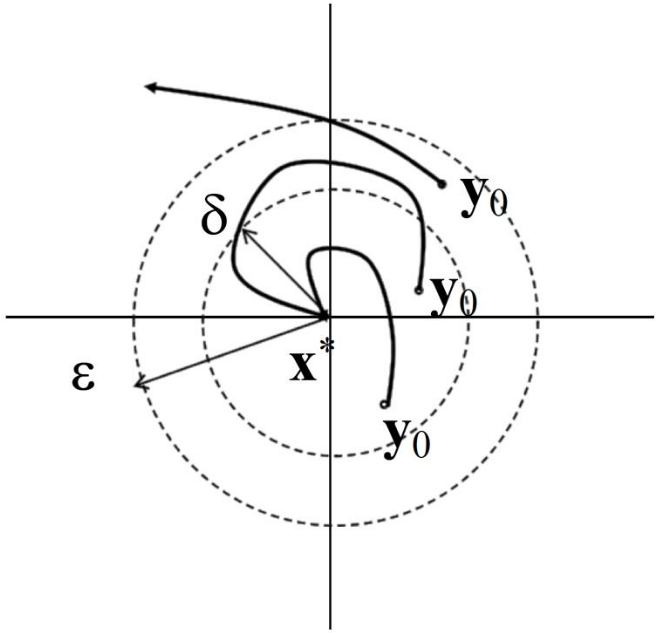
- **Definition**: An equilibrium point $\mathbf{x}^*$ is asymptotically stable if it is stable and $\beta > 0$ exists such that

$$\lim_{t \to \infty} \|\mathbf{x}(t) - \mathbf{x}^*\| = 0 \text{ whenever } \|\mathbf{x}(0) - \mathbf{x}^*\| < \beta$$

Picturing stability



Stable equilibrium



Asympotically stable equilibrium

- The solution cannot escape from a **stable** equilibrium
- The solution converges to the equilibrium point if it starts close enough to an **asymptotically stable** equilibrium

Epsilon-delta arguments

- Observe that the argument made within a definition like

$$\|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon \quad \text{for all} \quad t \geq 0 \quad \text{whenever} \quad \|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$$

takes the form of a game:

1. I give you a positive number ε that I am free to choose
2. you respond with a number δ that satisfies some condition
3. if you can find a number δ for any ε, you 'win'

- Many mathematical proofs and definitions are based on this kind of argument

Exponential stability

- **Definition**: An equilibrium point $\mathbf{x}^*$ is **exponentially stable** if $\mathbf{x}^*$ is asymptotically stable and there exist finite constants α, β, $\delta > 0$ such that

  $$\|\mathbf{x}(t) - \mathbf{x}^*\| < \alpha e^{-\beta t}\|\mathbf{x}(0) - \mathbf{x}^*\| \ \forall t \geq 0 \quad \text{whenever} \quad \|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$$

- As well as requiring that the solution is stable and converges to the equilibrium point (asymptotic stability), this also quantifies the **rate of convergence**

  i.e. how fast the solution flows to the equilibrium point

Flows in 2nd order linear systems

- Ultimately we will study flows around the equilibrium points of nonlinear ODE systems by examining **local linearizations** around those points
- Each flow has a **topology** (a shape) that falls into one of a number of distinct categories
- The flows in local linearizations can often be continuously distorted into flows that solve the nonlinear ODE systems
- It is useful to study the topologies of some example linear systems to understand what families of solutions look like

Uncoupled second order linear system

- Perhaps the simplest problem we can think of is

$$\frac{dx_1}{dt} = \alpha_1 x_1 \qquad \frac{dx_2}{dt} = \alpha_2 x_2 \qquad \text{solved by} \qquad x_1(t) = x_1(0)e^{\alpha_1 t} \qquad x_2(t) = x_2(0)e^{\alpha_2 t}$$

- These can be viewed as parametric equations that describe the shapes of curves in phase space

$$\left(x_1(t)\right)^{\alpha_2/\alpha_1} = \left(x_1(0)\right)^{\alpha_2/\alpha_1} e^{\alpha_2 t} = \frac{\left(x_1(0)\right)^{\alpha_2/\alpha_1}}{x_2(0)} x_2(t)$$

$$\Longrightarrow \quad x_2 = c x_1^{\alpha_2/\alpha_1}$$

Stability of the decoupled system

- Let's look more closely at the system

$$\frac{dx_1}{dt} = \alpha_1 x_1 \qquad \qquad x_1(t) = x_1(0)e^{\alpha_1 t}$$

solved by

$$\frac{dx_2}{dt} = \alpha_2 x_2 \qquad \qquad x_2(t) = x_2(0)e^{\alpha_2 t}$$

- The system has an equilibrium point at the origin (because it's an autonomous linear system)

- If $\alpha_1 < 0$ and $\alpha_2 < 0$, then the system is asymptotically stable, in fact **exponentially stable**

- If $\alpha_1 > 0$ or $\alpha_2 > 0$, then the system is **unstable**

Coupled second order linear system

- For a coupled linear system the analysis is slightly more complicated – consider

$$\frac{dx_1}{dt} = ax_1 + bx_2 \qquad \text{or} \qquad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \ \text{where } \mathbf{A} = \begin{bmatrix} a & d \\ c & b \end{bmatrix}$$
$$\frac{dx_2}{dt} = cx_1 + dx_2$$

- Here the solution for initial condition $\mathbf{x}(0)$ is $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0)$

- To simplify the matrix exponential term, diagonalize matrix $\mathbf{A}$ (assuming it is diagonalizable):

$$e^{t\mathbf{A}} = \mathbf{V}\mathrm{diag}\{e^{\lambda_i t}\}\mathbf{V}^{-1}$$

and rewrite the solution in terms of eigenvalues and eigenvectors

Eigenvalues

- Here the eigenvalues of $\mathbf{A}$ are found by solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{bmatrix} a-\lambda & d \\ c & b-\lambda \end{bmatrix} = \lambda^2 - (a+b)\lambda + (ab - cd) = 0$$

- This characteristic equation can also be written in terms of the **trace** and **determinant** of $\mathbf{A}$:

$$\lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$$

Recall that the trace of a matrix is the sum of its eigenvalues:

$$\mathrm{tr}(\mathbf{A}) = \lambda_1 + \lambda_2$$

and its determinant is the product of its eigenvalues:

$$\det(\mathbf{A}) = \lambda_1\lambda_2$$

Solving with eigenvectors and eigenvalues

- If the eigenvalues of $\mathbf{A}$ are real and distinct, then we can write any initial condition as a combination of the eigenvectors:

$$\mathbf{x}(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{V}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{V}^{-1}\mathbf{x}(0)$$

- Then, using $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0)$, we get

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{V}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{V}\mathrm{diag}\{e^{\lambda_i t}\}\mathbf{V}^{-1}\mathbf{V}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

- The solution is a linear combination of exponential transients with decay rates determined by the eigenvalues

Characteristics of linear system trajectories

- If $\mathrm{Re}(\lambda) < 0$, then the component of the solution along the corresponding eigenvector **decays to zero**
- If $\mathrm{Re}(\lambda) > 0$, then the component of the solution along the corresponding eigenvector **grows exponentially**
- If $\mathrm{Re}(\lambda) = 0$, then the component of the solution along the corresponding eigenvector **remains constant**
- If $\mathrm{Im}(\lambda) \neq 0$, then the solution orbits or **spirals** around the origin
- If $\mathrm{Im}(\lambda) = 0$, then the solution tends towards the eigenvector with the dominant eigenvalue

Coordinate transformation to normal form

Characteristic trajectory shapes in 2-D phase space are found using a coordinate transformation that puts $\mathbf{A}$ into a standard form:

- given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, define new coordinates $\mathbf{y} = \mathbf{K}^{-1}\mathbf{x}$
- this transforms the equations of motion to $\dot{\mathbf{y}} = \mathbf{K}\mathbf{A}\mathbf{K}^{-1}\mathbf{y}$
- if the eigenvalues of $\mathbf{A}$ are distinct and real, let $\mathbf{K} = \mathbf{V}$ then $\mathbf{V}\mathbf{A}\mathbf{V}^{-1} = \mathbf{D}$

$$\dot{\mathbf{y}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{y}$$

- if the eigenvalues of $\mathbf{A}$ are complex, $\lambda = a \pm jb$, then let $\mathbf{K} = \mathbf{V}'$:

$$\dot{\mathbf{y}} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mathbf{y} \quad \text{where} \quad \begin{array}{l} \mathbf{V}' = \begin{bmatrix} \mathbf{w} & \mathbf{u} \end{bmatrix}, \text{ and} \\ \mathbf{v} = \mathbf{u} \pm j\mathbf{w} \text{ are the eigenvectors} \end{array}$$

The transformed matrix is called the **normal form** of $\mathbf{A}$

The special case of a defective matrix

- If the eigenvalues of $\mathbf{A}$ are real but not distinct, (e.g. $\lambda_1 = \lambda_2$), then $\mathbf{A}$ may not have n linearly independent eigenvectors (i.e. $\mathbf{A}$ is **defective**)

  e.g. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has eigenvalue $\lambda = 1$ (multiplicity 2), eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- In this case we can compute a **generalised eigenvector** $\mathbf{v}_2$ using

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_2 = \mathbf{v}_1 \quad \Longrightarrow \quad (\mathbf{A} - \lambda_1 \mathbf{I})^2 \mathbf{v}_2 = (\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_1 = 0$$

  If λ_i has multiplicity m, it has m linearly independent generalised eigenvectors

- Defining the transformation matrix $\mathbf{V}' = [\mathbf{v}_1 \;\; \mathbf{v}_2]$ allows us to express $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{V}' \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} (\mathbf{V}')^{-1}$$

  This is called the **Jordan normal form** of $\mathbf{A}$

Saddle equilibrium points via normal forms

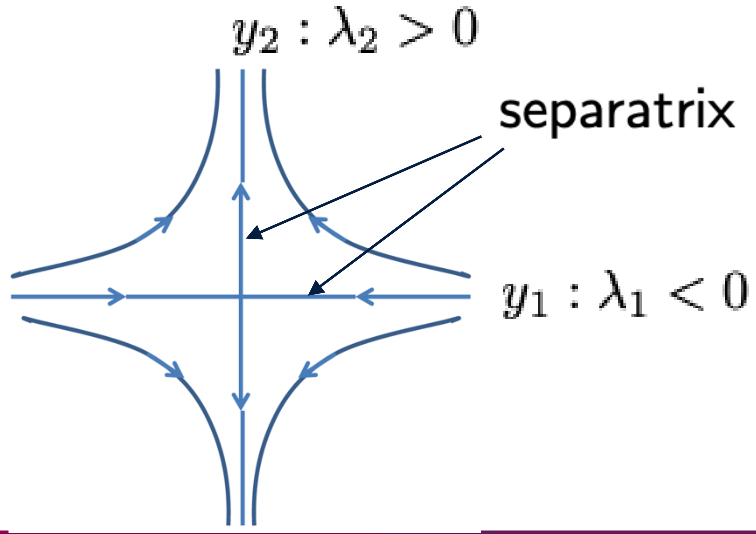
- If the eigenvalues of $\mathbf{A}$ are real and $\lambda_1\lambda_2 < 0$, then the equilibrium $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is unstable – this results in a **saddle point**

In transformed coordinates: $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

- The phase portrait has four asymptotes: two correspond to solutions that approach the origin as $t \to \infty$, and two as $t \to -\infty$

- These four trajectories are called **separatrices**

a saddle shape in the phase plane:



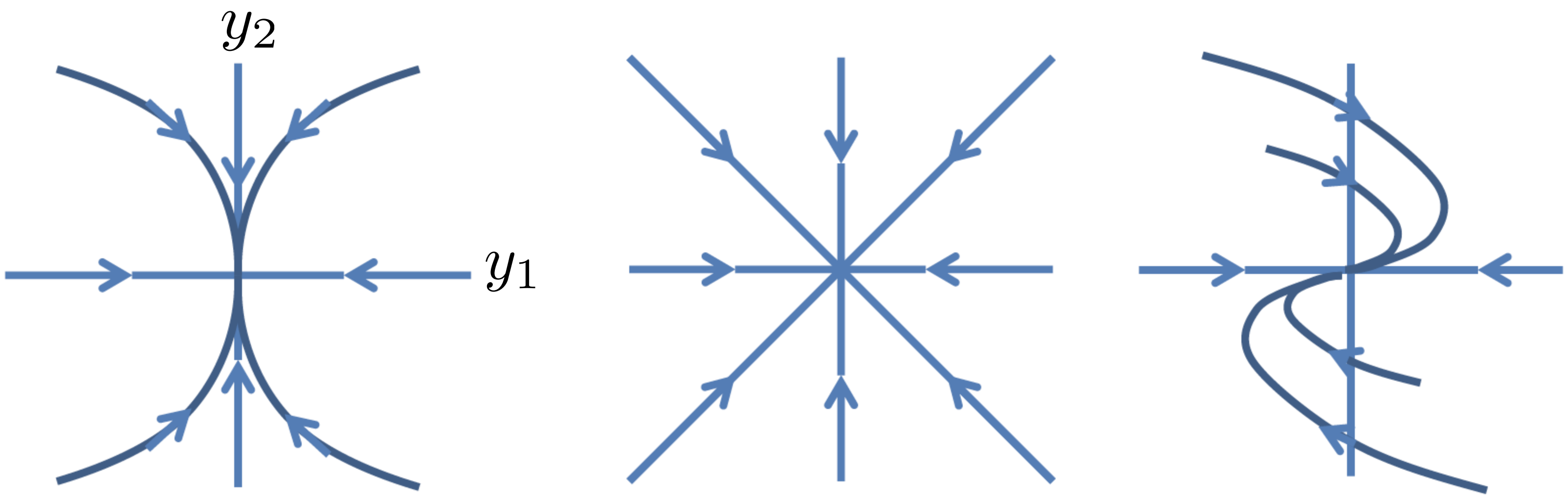
Stable equilibrium points via normal forms

- If the eigenvalues of $\mathbf{A}$ are real and both $\lambda_1 < 0$ and $\lambda_2 < 0$, then the equilibrium will be **stable**

- Three cases:
  - Eigenvalues distinct: $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$
  - Eigenvalues repeated but two eigenvectors: $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$
  - Eigenvalues repeated but $\mathbf{A}$ defective: $(\mathbf{V}')^{-1}\mathbf{A}\mathbf{V}' = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$

Stable and unstable flow shapes

- Consider the transformed coordinates: $\mathbf{y} = \mathbf{V}^{-1}\mathbf{x}$ or $\mathbf{y} = (\mathbf{V}')^{-1}\mathbf{x}$
- Stable solutions when both eigenvalues are real and negative:



standard case: $\lambda_1 < \lambda_2$ nondegenerate: $\lambda_1 = \lambda_2$ degenerate

- If both eigenvalues are positive then the solutions are unstable;
  flows in the phase plane look the same as above, but the arrows point in the opposite directions

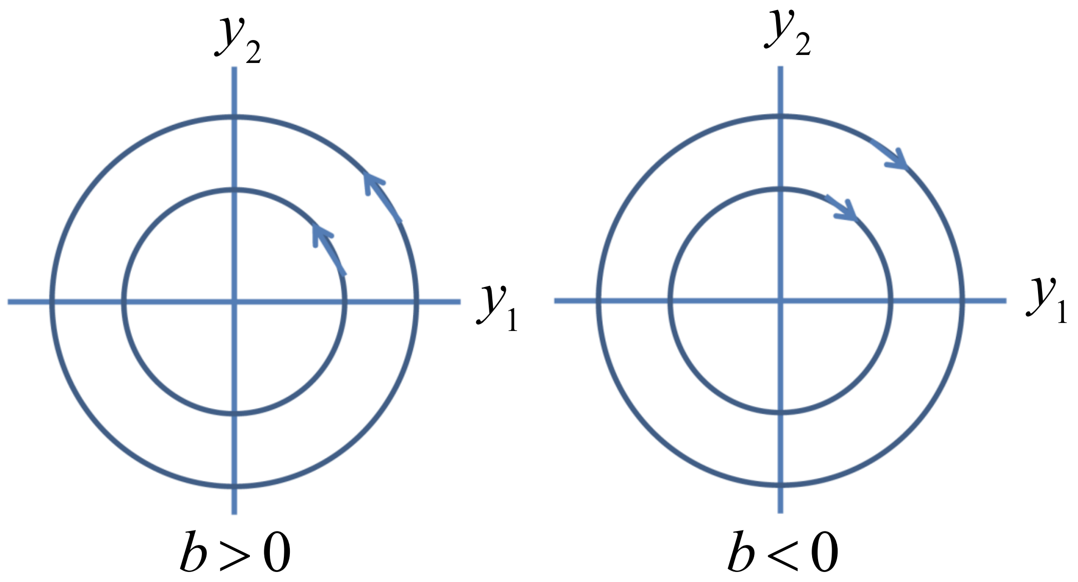
Centre equilibrium points via normal forms

- If the eigenvalues are purely imaginary (real part equal to zero), then the corresponding equilibrium point is marginally stable

- In this case $\lambda = 0 \pm jb$ and the transformation on slide 17 shows that the normal form is

$$(\mathbf{V}')^{-1}\mathbf{A}\mathbf{V}' = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$$

- Phase portraits:



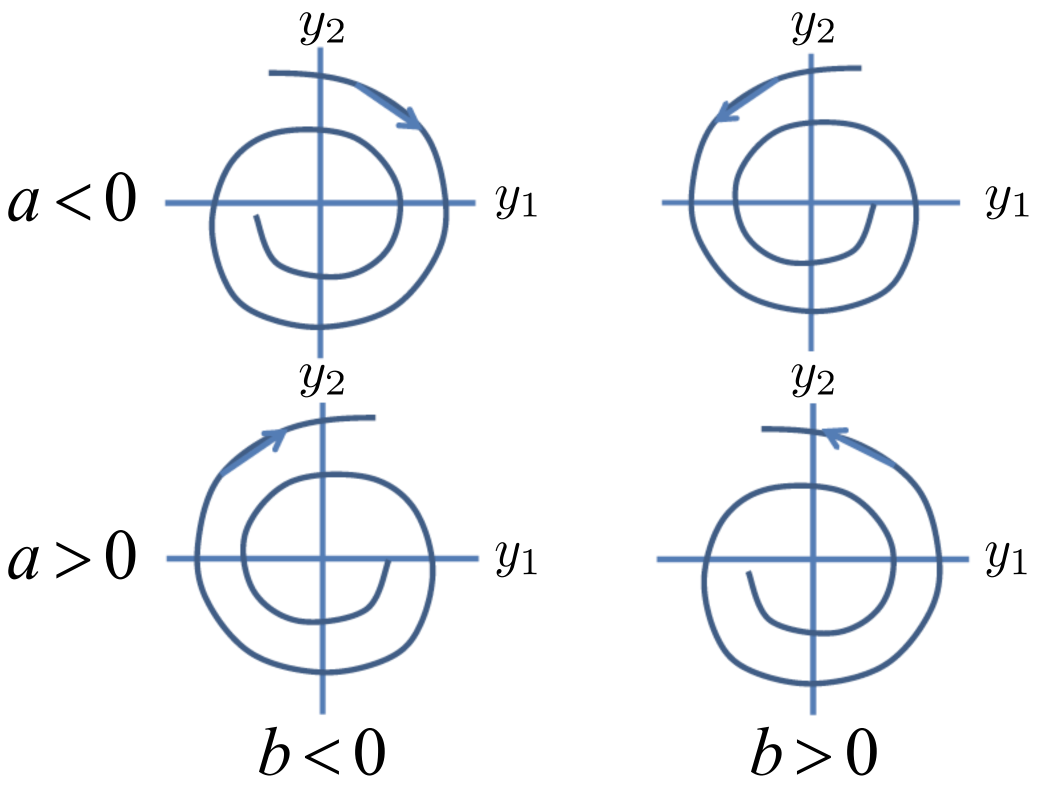
Stable and unstable spirals

- If the eigenvalues are $\lambda_1 = a + jb$, $\lambda_2 = a - jb$, then the normal form is

$$(\mathbf{V}')^{-1}\mathbf{AV}' = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

- Phase portraits are **spirals**

$a < 0$	:	spiral is stable
$a > 0$	:	spiral is unstable
$b > 0$	:	spiral is clockwise
$b < 0$	:	spiral is anticlockwise



Phase space equation summary

- We have sketched how solutions behave in the phase plane using the normal forms of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$:

$$\dot{\mathbf{y}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{y} \implies \mathbf{y}(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \mathbf{y}_0$$

real; complete eigenvector set

$$\dot{\mathbf{y}} = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \mathbf{y} \implies \mathbf{y}(t) = e^{\lambda_1 t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{y}_0$$

real; degenerate

$$\dot{\mathbf{y}} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mathbf{y} \implies \mathbf{y}(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{y}_0$$

complex

Example of coordinate change

- Consider the system $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \mathbf{x}$
- Eigenvalues from characteristic equation: $\lambda^2 + \lambda - 2 = 0 \implies \lambda \in \{1, -2\}$
- Eigenvectors: $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\implies \mathbf{A} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

- Define new coordinates $\mathbf{y} = \mathbf{V}^{-1}\mathbf{x}$, then the transformed coordinate axes are:

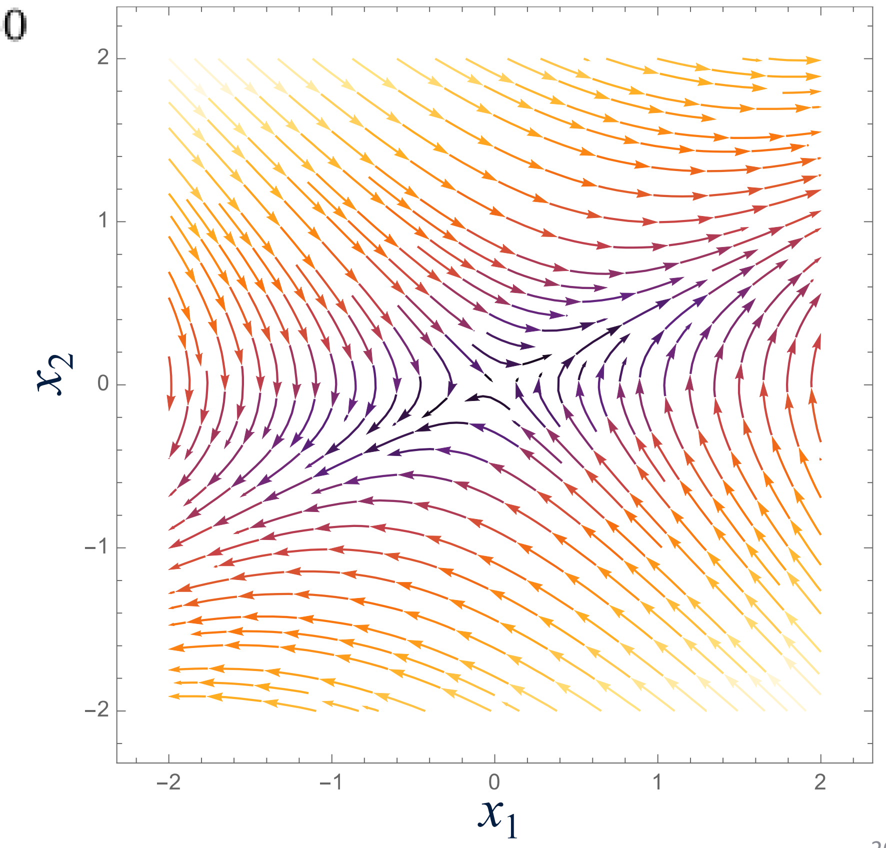
$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (unstable), and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (stable)

Example continued

- Transformed system: $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \mathbf{x}$

- Solution: $\mathbf{x}(t) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \mathbf{x}_0$

$$\mathbf{x}(t) = \frac{1}{3} \begin{bmatrix} e^{-2t} + 2e^{t} & -2e^{-2t} + 2e^{t} \\ -e^{-2t} + e^{t} & 2e^{-2t} + e^{t} \end{bmatrix} \mathbf{x}_0$$



Linearization of nonlinear systems

- Consider the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with equilibrium at $\mathbf{x}^*$
- Let $\mathbf{x} = \mathbf{x}^* + \mathbf{w}$ and assume that $\mathbf{f}$ is differentiable.
  Then the Taylor expansion of the ith entry, f_i, of $\mathbf{f}$ gives

$$f_i(\mathbf{x}^* + \mathbf{w}) = f_i(\mathbf{x}^*) + \sum_{j=1}^{n} \left(\left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{x}^*} w_j + O(|w_j|^2) \right)$$

- Noting that equilibrium is independent of time by definition

$$\dot{\mathbf{x}} = \dot{\mathbf{w}} = D\mathbf{f}(\mathbf{x}^*)\mathbf{w} + O(\|\mathbf{w}\|^2)$$

- The matrix $D\mathbf{f}(\mathbf{x})$ is called the **Jacobian** of the vector-valued function $\mathbf{f}$, with ijth entry $(D\mathbf{f})_{ij} = \partial f_i / \partial x_j$

Hyperbolic equilibria

Definition (hyperbolic equilibrium): If $\mathbf{x}^*$ is an equilibrium of system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then $\mathbf{x}^*$ is called a *hyperbolic fixed point* if all eigenvalues of the Jacobian $D\mathbf{f}(\mathbf{x}^*)$ have nonzero real parts

- This leads to an important **theorem**: if an equilibrium point is a hyperbolic fixed point and all the eigenvalues of the Jacobian have negative real parts, then the equilibrium solution $\mathbf{x} = \mathbf{x}^*$ is asymptotically stable

- Note that this proves asymptotic stability but does not say anything about the size of the region of stability
  – this depends on the size of δ from the game on slide 8

- In lecture 3 we will see how linearizations near equilibrium points can be used to get information about nonlinear systems

Example: Duffing oscillator

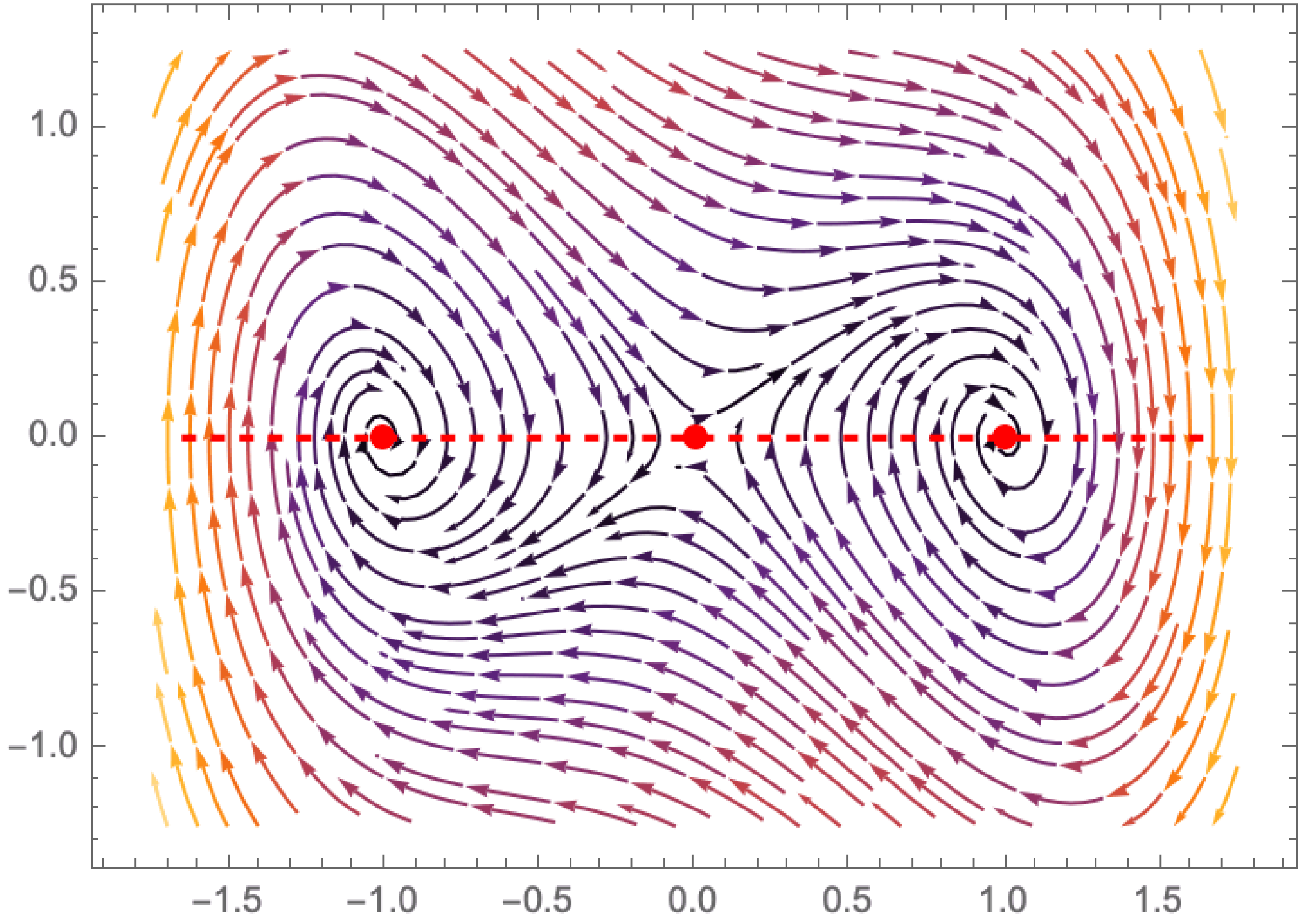
The Duffing oscillator is described for $\gamma \geq 0$ by

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = x - x^3 - \gamma y$$

- equilibria: $(x^*, y^*) = (0,0)$ and $(\pm 1, 0)$
- Jacobian: $D\mathbf{f} = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -\gamma \end{bmatrix}$
- at $(0,0)$: $\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2+4}}{2} \implies$ unstable

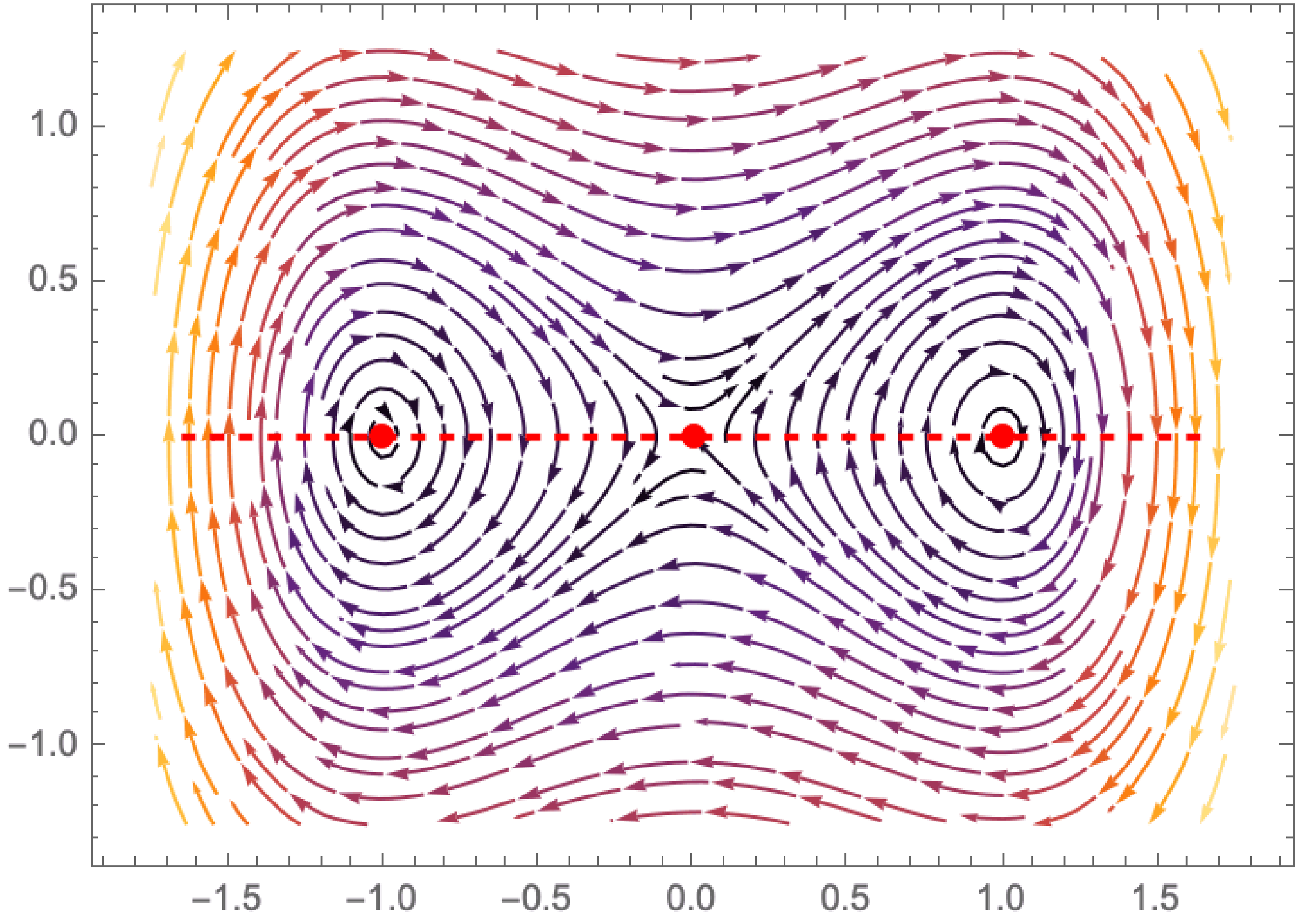
  at $(\pm 1, 0)$: $\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2-8}}{2} \implies$ asymptotically stable if $\gamma > 0$
- If $\gamma = 0$, $(\pm 1, 0)$ is a centre $\implies$ local linearization inconclusive

Example: Duffing oscillator



Phase plane of the Duffing oscillator with $\gamma = 0.5$

Example: Duffing oscillator



Phase plane of the Duffing oscillator with $\gamma = 0$

Questions?